A note on the Chevalley–Eilenberg cohomology for the Galilei and Poincaré algebras

A note on the Chevalley–Eilenberg cohomology for the Galilei and Poincaré algebras

Sotirios Bonanos[1] **and Joaquim Gomis**[2]

[1] Institute of Nuclear Physics, NCSR Demokritos, 15310 Aghia Paraskevi, Attiki, Greece
[2] Departament d'Estructura i Constituents de la Matèria and ICCUB, Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain

E-mail: sbonano@inp.demokritos.gr and gomis@ecm.ub.es

Abstract
We construct in a systematic way the complete Chevalley–Eilenberg cohomology at form degrees 2, 3 and 4 for the Galilei and Poincaré groups. The corresponding non-trivial forms belong to certain representations of the spatial rotation (Lorentz) group. In the case of 2-forms they give all possible central and non-central extensions of the Galilei group (and all non-central extensions of the Poincaré group). The procedure developed in this paper can be applied to any spacetime symmetry group.

PACS numbers: 02.20.Qs, 02.70.Wz, 11.30.Cp

1. Introduction and results

Physical theories are characterized by their symmetries. The basic symmetry is that of the underlying space: the transformations that are implied by the homogeneity and isotropy of space. Physical theories respect these kinematical symmetries and exhibit additional dynamical symmetries. Among the possible kinematical groups, which have been classified in [1], the Poincaré and Galilei groups are the better known ones. Here we will study the central and non-central extensions of these two groups, obtaining extended symmetries which can be used to construct new physical theories or to interpret group theoretically certain symmetry properties of known theories.

It is well known that the Galilei group in d+1 spacetime dimensions admits a central extension [2], while in the special case of 2+1 dimensions, it admits an exotic, two-fold central extension [3–5]. This exotic Galilei group has attracted attention in the context of non-commutative geometry and in condensed matter physics [6–9], see also [10] for further references. In the particular case of 1+1 dimensions the Galilei algebra has two central

extensions and the Poincaré algebra one[3] [11]. The Poincaré group has no central extensions in $d+1$ $(d > 1)$ spacetime dimensions; however it has a non-central antisymmetric tensor extension, which has an important physical application: it describes the motion of a relativistic particle in a constant electromagnetic field [15–17], see also [18]. Tensor central extensions in Euclidean R^{2n} or R^{4n} spaces with U(n) symmetry group were constructed in [19, 20].

The extensions of a group are related to a non-trivial Chevalley–Eilenberg (CE) cohomology of the corresponding unextended Lie algebras, for a review see [21]. In this paper we compute in a systematic, almost algorithmic, way the most general CE cohomology groups at form degrees 2, 3 and 4. As we will see the non-trivial forms belong to different representations of the subgroup of spatial rotations (Lorentz) of the Galilei (Poincaré) group.

In all spacetime algebras we can separate the generators into 'translations' and 'rotations'. The 'rotations' generators are a subgroup of the automorphism group and constitute a normal subgroup of every extended group. In the case of the Galilei group we consider as 'translations' the space and time translations and the Galilean boosts, while the 'rotations' are the ordinary spatial rotations. For the Poincaré group 'translations' are the spacetime translations and the 'rotations' are the Lorentz transformations.

The first extension, central and non-central, of the unextended Galilei algebra is obtained by calculating all possible non-trivial closed 2-forms of the subgroup of 'translations'. The forms are closed with respect to the exterior differential operator d. These forms can be grouped into different representations of the rotation group: a vector, an antisymmetric tensor and a symmetric tensor. (In the case of Poincaré there is only an antisymmetric tensor representation.) The trace of the symmetric tensor corresponds to the well-known central extension of the Galilei group [2]. In the case of 2+1 dimensions the antisymmetric tensor becomes a pseudoscalar representation which gives rise to the second central extension.

The complete extended algebra is constructed from the original algebra and the extensions by incorporating their transformation properties under rotations, which is equivalent to replacing the exterior differential operator d by the corresponding 'covariant' operator $d + M\wedge$, M being the zero-curvature connection associated with the 'rotations': $dM + M \wedge M = 0$.

As we will see, once we have an extended algebra, we can further extend it by applying the previous procedure to the new set of translations that now includes the extended generators. In this way we obtain new extensions with some generators belonging to higher dimensional representations of the corresponding 'rotations' group. The physical significance of these higher dimensional extensions should be clarified. At this level the generators of the first and second level extensions couple non-trivially to each other. This procedure for the case of the Galilei and Poincaré groups does not end.

We have also studied the first level CE cohomology at degrees 3 and 4. As in the case of 2-forms, the non-trivial 3- or 4-forms obtained are grouped into representations of the 'rotations' group. The non-invariant 2-form 'potentials' associated with these closed 3-forms could be used to construct non-relativistic string theories whose Lagrangian will be invariant up to a total derivative under the Galilei (Poincaré) group.

For the Galilei group these string theories will be different from the non-relativistic string theory introduced in [22–24] because the invariance group of this string theory is the Galilean string group which is obtained by a stringy contraction of the Poincaré group [25] and not by the ordinary contraction that leads to the ordinary Galilei algebra. In the case of the Poincaré group the forms can be used to construct Wess–Zumino (WZ) terms for relativistic string

---

[3] These 1+1 extended algebras have been used to study several problems of gravity and Moyal quantization, see for example [12–14].

theories coupled to external backgrounds. Finally, we note that the non-trivial closed 4-forms obtained here could be used to construct WZ actions of non-relativistic and relativistic branes.

The procedure to compute CE cohomology groups employed in this paper is described in the appendix and can be applied to any spacetime symmetry group. The results are given for the case of 3+1 dimensions, but this procedure, mutatis mutandis, can be used in any number of space dimensions. The application to super spacetime groups with odd generators will be done in a forthcoming paper.

The organization of the paper is as follows. In section 2, we give our notation and conventions and study the first level extension of the Galilei group. In section 3, we analyze the second level extension. In section 4, we give the results of the first level CE cohomology at form degrees 3 and 4, while in section 5 we give the corresponding results for the Poincaré group. Finally, in an appendix, we describe the 'algorithm' used for obtaining these extensions. The calculations make use of the first author's Mathematica package EDC (exterior differential calculus) [26].

2. Galilei group in 3+1 dimensions

The generators of the unextended Galilei algebra are the Hamiltonian, H, the spatial translations P_a, the boots K_a and the rotations M_{ab}. The algebra is given by[4]

$$
\begin{aligned}
&[M_{ab}, M_{cd}] = -i\eta_{b[c}M_{ad]} + i\eta_{a[c}M_{bd]}, \\
&[K_a, M_{cd}] = -i\eta_{a[c}K_{d]}, \\
&[P_a, M_{bc}] = -i\eta_{a[b}P_{c]}, \\
&[H, M_{ab}] = 0, \\
&[H, K_a] = +iP_a, \\
&[H, P_a] = 0, \\
&[K_a, K_b] = 0, \\
&[P_a, K_b] = 0, \\
&[P_a, P_b] = 0,
\end{aligned}
\tag{2.1}
$$

where η_{ab} is the euclidean metric of three-dimensional space, $a, b = 1, 2, 3$.

In order to construct the extensions it is very useful to introduce the left invariant Maurer–Cartan (MC) form, defined by

$$
\Omega = -ig^{-1}\,\mathrm{d}g,
\tag{2.2}
$$

where g represents a general element of the Galilei group. The MC form satisfies the MC equation

$$
\mathrm{d}\Omega + i\Omega \wedge \Omega = 0.
\tag{2.3}
$$

In components the MC 1-form is written, for a generic Lie algebra, as

$$
\Omega = X_A \mathcal{X}^A,
\tag{2.4}
$$

where X_A are the generators of the Lie algebra satisfying

$$
[X_B, X_C] = i f^A{}_{BC} X_A
\tag{2.5}
$$

---

[4] Although the algebra is real and the imaginary units can be made to disappear by replacing all generators G by iG', we prefer to leave the i's in the equations because then we can interpret the generators as Hermitian operators.

and $\mathcal{X}^{\mathcal{A}}$ are corresponding 1-forms. (Throughout this paper we use the same capital letters in plain and calligraphic font to denote generators and associated 1-forms.) The MC equation (2.3) implies that the 1-forms $\mathcal{X}^{\mathcal{A}}$ satisfy

$$\mathrm{d}\mathcal{X}^A = \tfrac{1}{2} f^A{}_{BC} \mathcal{X}^B \wedge \mathcal{X}^C. \tag{2.6}$$

For the Galilei case, the MC 1-form (2.4) becomes

$$\Omega = H\mathcal{H} + P_a\mathcal{P}^a + K_a\mathcal{K}^a + \tfrac{1}{2}M_{ab}\mathcal{M}^{ab}, \tag{2.7}$$

while the MC equation (2.3) in components is

$$\begin{aligned}
&\mathrm{d}\mathcal{H} = 0, \\
&\mathrm{d}\mathcal{P}^a + \mathcal{P}^c \wedge \mathcal{M}_c{}^a - \mathcal{H} \wedge \mathcal{K}^a = 0, \\
&\mathrm{d}\mathcal{K}^a + \mathcal{K}^c \wedge \mathcal{M}_c{}^a = 0, \\
&\mathrm{d}\mathcal{M}^{ab} + \mathcal{M}^{ac} \wedge \mathcal{M}_c{}^b = 0.
\end{aligned} \tag{2.8}$$

In order to start our procedure for the cohomological analysis we need to define the 'translations' generators, which we take to be the space and time translations and the boosts, i.e., H, P_a, K_a. The MC equations for these generators are obtained by putting $\mathcal{M}^{ab} \to 0$ in 2.8.

The most general closed invariant 2-form which cannot be written as the differential of an invariant 1-form is then found to be

$$\Omega_2 = f_a\mathcal{H} \wedge \mathcal{P}^a + f_{[ab]}\mathcal{K}^a \wedge \mathcal{K}^b + f_{(ab)}\mathcal{K}^a \wedge \mathcal{P}^b, \tag{2.9}$$

where the constant parameters are a vector f_a, a second rank antisymmetric tensor $f_{[ab]}$ and a second rank symmetric tensor $f_{(ab)}$.[5] Therefore we find that the non-trivial 2-forms belong to a vector, a symmetric and an antisymmetric tensor representation of the rotation group. The 1-form 'potentials' associated with these 2-forms are denoted by

$$\mathcal{Z}^a, \quad \mathcal{Z}^{[ab]}, \quad \mathcal{Z}^{(ab)}, \tag{2.10}$$

and satisfy the MC equations

$$\begin{aligned}
&\mathrm{d}\mathcal{Z}^a = \mathcal{H} \wedge \mathcal{P}^a, \\
&\mathrm{d}\mathcal{Z}^{[ab]} = \mathcal{K}^a \wedge \mathcal{K}^b, \\
&\mathrm{d}\mathcal{Z}^{(ab)} = \mathcal{K}^a \wedge \mathcal{P}^b + \mathcal{K}^b \wedge \mathcal{P}^a.
\end{aligned} \tag{2.11}$$

From these expressions we can obtain the algebra of the corresponding generators, denoted by

$$Z_a, \quad Z_{[ab]}, \quad Z_{(ab)}. \tag{2.12}$$

We find

$$\begin{aligned}
&[K_a, K_b] = +\mathrm{i}Z_{[ab]}, \\
&[P_a, K_b] = -\mathrm{i}Z_{(ab)}, \\
&[H, P_a] = +\mathrm{i}Z_a.
\end{aligned} \tag{2.13}$$

The extension Z_a has been used to describe the motion of a particle in a constant field, $Z_{[ab]}$ to study the motion of a particle in a non-commutative space, while $Z_{(ab)}$ appears in the description of a particle with mass anisotropy. The ordinary central extension of the Galilei group corresponds to the trace part of $Z_{(ab)}$. In 2+1 dimensions, the second (exotic) central extension corresponds to the single independent component of the antisymmetric tensor $Z_{[ab]}$,

---

[5] The calculation proceeds in several distinct steps, as outlined in the appendix.

while in 1+1 dimensions the single-component vector extension Z_a gives the extra central charge of Galilei [11].

With the rotations included, the extended set of MC 1-forms satisfies the equations

$$
\begin{aligned}
&\mathrm{d}\mathcal{H} = 0, \\
&\mathrm{d}\mathcal{P}^a = -\mathcal{P}^c \wedge \mathcal{M}_c{}^a + \mathcal{H} \wedge \mathcal{K}^a, \\
&\mathrm{d}\mathcal{K}^a = -\mathcal{K}^c \wedge \mathcal{M}_c{}^a, \\
&\mathrm{d}\mathcal{M}^{ab} = -\mathcal{M}^{ac} \wedge \mathcal{M}_c{}^b, \\
&\mathrm{d}\mathcal{Z}^a = -\mathcal{Z}^c \wedge \mathcal{M}_c{}^a + \mathcal{H} \wedge \mathcal{P}^a, \\
&\mathrm{d}\mathcal{Z}^{[ab]} = -\mathcal{Z}^{[ac]} \wedge \mathcal{M}_c{}^b - \mathcal{M}^a{}_c \wedge \mathcal{Z}^{[cb]} + \mathcal{K}^a \wedge \mathcal{K}^b, \\
&\mathrm{d}\mathcal{Z}^{(ab)} = -\mathcal{Z}^{(ac)} \wedge \mathcal{M}_c{}^b - \mathcal{M}^a{}_c \wedge \mathcal{Z}^{(cb)} + \mathcal{K}^a \wedge \mathcal{P}^b + \mathcal{K}^b \wedge \mathcal{P}^a.
\end{aligned}
\tag{2.14}
$$

2.1. Explicit parametrization

Here we will obtain an explicit parametrization for all MC 1-forms in terms of differentials of functions, so that equations (2.14) are satisfied. We first obtain expressions for the unextended MC 1-forms without rotations. Locally, we can parametrize a general element of the unextended group by

$$
g = \mathrm{e}^{\mathrm{i}Hx^0}\, \mathrm{e}^{\mathrm{i}P_a x^a}\, \mathrm{e}^{\mathrm{i}K_a v^a}.
\tag{2.15}
$$

The MC 1-form (2.2) can be computed directly from its definition using the Baker–Campbell–Hausdorff formula. The components of the MC 1-form (2.4) can be computed explicitly using Mathematica and EDC [26], if we substitute a matrix representation for the generators, for example the adjoint representation. The result is

$$
\begin{aligned}
&\mathcal{H} = \mathrm{d}x^0, \\
&\mathcal{P}^a = \mathrm{d}x^a - v^a\, \mathrm{d}x^0, \\
&\mathcal{K}^a = \mathrm{d}v^a.
\end{aligned}
\tag{2.16}
$$

For the extended group, instead of following the same procedure starting with the general element

$$
g = \mathrm{e}^{\mathrm{i}Hx^0}\, \mathrm{e}^{\mathrm{i}P_a x^a}\, \mathrm{e}^{\mathrm{i}K_a v^a}\, \mathrm{e}^{\mathrm{i}Z_a k^a}\, \mathrm{e}^{\mathrm{i}Z_{(ab)}k^{(ab)}}\, \mathrm{e}^{\mathrm{i}Z_{[ab]}k^{[ab]}},
\tag{2.17}
$$

it is much easier to use the known parametrization for the 1-forms $\mathcal{H}$, $\mathcal{P}^a$, $\mathcal{K}^a$ on the right-hand side of (2.11) and integrate. The result is that the new MC 1-forms, modulo exact 1-forms, can be written as

$$
\begin{aligned}
&\mathcal{Z}^a = \mathrm{d}k^a - x^a\, \mathrm{d}x^0, \\
&\mathcal{Z}^{[ab]} = \mathrm{d}k^{[ab]} + \tfrac{1}{2}(v^a\, \mathrm{d}v^b - v^b\, \mathrm{d}v^a), \\
&\mathcal{Z}^{(ab)} = \mathrm{d}k^{(ab)} + v^a\, \mathrm{d}(x^b - v^b x^0) + v^b\, \mathrm{d}(x^a - v^a x^0),
\end{aligned}
\tag{2.18}
$$

where k^a, $k^{[ab]}$, $k^{(ab)}$ are new functions—the group parameters associated with the new generators in (2.17). These 1-forms can be used to construct non-relativistic particle Lagrangians. The study of these Lagrangians and their physical implications will be considered elsewhere. Here let us note that the familiar free non-relativistic massive particle is obtained by taking the trace of $\mathcal{Z}^{(ab)}$ and coincides with the one obtained with the method of nonlinear realizations [27], see for example [28].

If we want to have the explicit expressions of the MC 1-forms when we include rotations, the right-hand side of all vector and tensor expressions given above must be multiplied by an appropriate rotation matrix for each index:

$$
\mathcal{P}^a = U^{-1a}{}_b(\mathrm{d}x^b + \cdots), \qquad \mathcal{Z}^{ab} = U^{-1a}{}_p U^{-1b}{}_q(\mathrm{d}k^{pq} + \cdots),
\tag{2.19}
$$

where the rotation MC 1-forms have the explicit representation $\mathcal{M}^a{}_b = U^{-1a}{}_c d[U^c{}_b]$.

3. Second level extensions

One can obtain further extensions of the Galilei group corresponding to non-trivial 2-forms in higher dimensional representations of the rotation group. In order to find them we follow the same procedure as in the last section, this time taking as 'translations' the 19 1-forms

$$\mathcal{H}, \quad \mathcal{P}^a, \quad \mathcal{K}^a, \quad \mathcal{Z}^a, \quad \mathcal{Z}^{[ab]}, \quad \mathcal{Z}^{(ab)}. \tag{3.1}$$

We obtain 33 new non-trivial closed 2-forms which are the components of the following tensors:

$$
\begin{aligned}
&\mathcal{H} \wedge \mathcal{Z}^a, \\
&\mathcal{H} \wedge \mathcal{Z}^{(ab)} + \mathcal{K}^a \wedge \mathcal{Z}^b + \mathcal{K}^b \wedge \mathcal{Z}^a, \\
&\mathcal{P}^a \wedge \mathcal{P}^b + \mathcal{K}^a \wedge \mathcal{Z}^b - \mathcal{K}^b \wedge \mathcal{Z}^a, \\
&\mathcal{H} \wedge \mathcal{Z}^{[ab]} + \mathcal{K}^a \wedge \mathcal{P}^b - \mathcal{K}^b \wedge \mathcal{P}^a, \\
&\mathcal{K}^a \wedge \mathcal{Z}^{(bc)} + \mathcal{K}^b \wedge \mathcal{Z}^{(ca)} + \mathcal{K}^c \wedge \mathcal{Z}^{(ab)}, \\
&2\mathcal{Z}^{[ab]} \wedge \mathcal{K}^c - \mathcal{Z}^{[bc]} \wedge \mathcal{K}^a - \mathcal{Z}^{[ca]} \wedge \mathcal{K}^b,
\end{aligned}
\tag{3.2}
$$

the latter being a tensor antisymmetric in the first two indices (ab) whose totally antisymmetric part $\epsilon_{abc}\mathcal{Z}^{[ab]} \wedge \mathcal{K}^c$ vanishes.

If we use the kernel symbol $\mathcal{Y}$ for the 1-form tensor 'potentials' associated with these 2-forms and include the effect of rotations, we find that they satisfy the equations

$$
\begin{aligned}
d\mathcal{Y}^a ={}& -\mathcal{Y}^c \wedge \mathcal{M}_c{}^a + \mathcal{H} \wedge \mathcal{Z}^a, \\
d\mathcal{Y}^{(ab)} ={}& -\mathcal{Y}^{(ac)} \wedge \mathcal{M}_c{}^b - \mathcal{M}^a{}_c \wedge \mathcal{Y}^{(cb)} + \mathcal{H} \wedge \mathcal{Z}^{(ab)} + \mathcal{K}^a \wedge \mathcal{Z}^b + \mathcal{K}^b \wedge \mathcal{Z}^a, \\
d\mathcal{Y}_1{}^{[ab]} ={}& -\mathcal{Y}_1{}^{[ac]} \wedge \mathcal{M}_c{}^b - \mathcal{M}^a{}_c \wedge \mathcal{Y}_1{}^{[cb]} + \mathcal{P}^a \wedge \mathcal{P}^b + \mathcal{K}^a \wedge \mathcal{Z}^b - \mathcal{K}^b \wedge \mathcal{Z}^a, \\
d\mathcal{Y}_2{}^{[ab]} ={}& -\mathcal{Y}_2{}^{[ac]} \wedge \mathcal{M}_c{}^b - \mathcal{M}^a{}_c \wedge \mathcal{Y}_2{}^{[cb]} + \mathcal{H} \wedge \mathcal{Z}^{[ab]} + \mathcal{K}^a \wedge \mathcal{P}^b - \mathcal{K}^b \wedge \mathcal{P}^a, \\
d\mathcal{Y}^{(abc)} ={}& -\mathcal{Y}^{(abs)} \wedge \mathcal{M}_s{}^c - \mathcal{Y}^{(bcs)} \wedge \mathcal{M}_s{}^a - \mathcal{Y}^{(cas)} \wedge \mathcal{M}_s{}^b + \mathcal{K}^a \wedge \mathcal{Z}^{(bc)} \\
&+ \mathcal{K}^b \wedge \mathcal{Z}^{(ca)} + \mathcal{K}^c \wedge \mathcal{Z}^{(ab)}, \\
d\mathcal{Y}_3{}^{[ab]c} ={}& -\mathcal{M}^a{}_s \wedge \mathcal{Y}_3{}^{[sb]c} - \mathcal{M}^b{}_s \wedge \mathcal{Y}_3{}^{[as]c} - \mathcal{M}^c{}_s \wedge \mathcal{Y}_3{}^{[ab]s} + 2\mathcal{Z}^{[ab]} \\
&\wedge \mathcal{K}^c - \mathcal{Z}^{[bc]} \wedge \mathcal{K}^a - \mathcal{Z}^{[ca]} \wedge \mathcal{K}^b,
\end{aligned}
\tag{3.3}
$$

where the symmetries of the $\mathcal{Y}$ tensors are as indicated by round or square brackets, while the totally antisymmetric part of $\mathcal{Y}_3{}^{[ab]c}$ vanishes. The corresponding new algebra generators will be written as

$$Y_a, \quad Y_{(ab)}, \quad Y^1{}_{[ab]}, \quad Y^2{}_{[ab]}, \quad Y_{(abc)}, \quad Y^3{}_{[ab]c}. \tag{3.4}$$

We observe that, at this level the generators of the extensions couple non-trivially to each other, for example

$$[K_a, Z_{(bc)}] = iY_{(abc)}, \tag{3.5}$$

while the P_a no longer commute

$$[P_a, P_b] = iY^1{}_{[ab]}. \tag{3.6}$$

The generator $Y^1{}_{[ab]}$ can be used to describe the motion of non-relativistic particles in a constant magnetic background.

We could complete the algebra by introducing the transformations of the new generators under rotations. For the case of the Galilei group this extension procedure does not end. The physical interpretation of these new $\mathcal{Y}$ terms remains to be understood.

4. CE cohomology at degrees 3 and 4

In this section we study the most general non-trivial closed 3-forms and 4-forms. Our procedure gives 18 closed non-trivial 3-forms, which, as in the case of 2-forms studied above, can be grouped under different representations of the rotation group, resulting in

- a symmetric tensor $\mathcal{H} \wedge (\mathcal{K}^a \wedge \mathcal{P}^b + \mathcal{K}^b \wedge \mathcal{P}^a)$;
- an antisymmetric tensor $\mathcal{H} \wedge \mathcal{P}^a \wedge \mathcal{P}^b$;
- a pseudoscalar $\epsilon_{abc}\mathcal{K}^a \wedge \mathcal{K}^b \wedge \mathcal{K}^c$;
- and the following third rank tensor, antisymmetric in its first two indices $[ab]$, whose totally antisymmetric part vanishes

$$2\,\mathcal{K}^a \wedge \mathcal{K}^b \wedge \mathcal{P}^c - \mathcal{K}^b \wedge \mathcal{K}^c \wedge \mathcal{P}^a - \mathcal{K}^c \wedge \mathcal{K}^a \wedge \mathcal{P}^b. \tag{4.1}$$

These closed 3-forms imply the existence of corresponding 2-form tensor potentials[6]. In terms of the explicit parametrization of group elements introduced in (2.16), (2.18), we can write these 2-form potentials (modulo exact of 2-forms) as

$$-(v^a \, \mathrm{d}x^0 \wedge \mathrm{d}x^b + v^b \, \mathrm{d}x^0 \wedge \mathrm{d}x^a), \qquad -\tfrac{1}{2}(x^a \, \mathrm{d}x^0 \wedge \mathrm{d}x^b - x^b \, \mathrm{d}x^0 \wedge \mathrm{d}x^a),$$

$$\epsilon_{abc}v^a \, \mathrm{d}v^b \wedge \mathrm{d}v^c, \qquad vvx(a, b, c) - \tfrac{1}{2}vvx(b, c, a) - \tfrac{1}{2}vvx(c, a, b), \tag{4.2}$$

where $vvx(a, b, c)$ stands for $(v^a \, \mathrm{d}v^b - v^b \, \mathrm{d}v^a) \wedge (\mathrm{d}x^c - v^c \, \mathrm{d}x^0)$.

These 2-forms could be used to construct non-relativistic string theories whose Lagrangian is invariant up to an exact form under the Galilei group. As mentioned in the Introduction, these string theories will be different from the non-relativistic string theory introduced in [22–24].

Finally, applying our procedure at degree 4 we obtain 19 non-trivial closed 4-forms which are the components of the following tensors:

$$\begin{aligned}
&\mathcal{H} \wedge \mathcal{K}^a \wedge \mathcal{P}^b \wedge \mathcal{P}^c, \\
&\mathcal{H} \wedge \epsilon_{abc}\mathcal{P}^a \wedge \mathcal{P}^b \wedge \mathcal{P}^c, \\
&\mathcal{P}^d \wedge \epsilon_{abc}\mathcal{K}^a \wedge \mathcal{K}^b \wedge \mathcal{K}^c, \\
&\mathcal{P}^a \wedge \mathcal{P}^b \wedge \mathcal{K}^c \wedge \mathcal{K}^d + \mathcal{K}^a \wedge \mathcal{K}^b \wedge \mathcal{P}^c \wedge \mathcal{P}^d.
\end{aligned} \tag{4.3}$$

The corresponding non-trivial forms could be used for constructing WZ terms of non-relativistic branes. In higher dimensions, the terms containing ϵ_{abc} would give totally antisymmetric third rank tensors.

5. Poincaré extensions

It is known that the Poincaré group admits no central extensions in $d > 1$ dimensions. However it has an antisymmetric tensor non-central extension [15–18]. Here we apply our procedure to the Poincaré group and obtain all non-central extensions. We will use the same notation as in the previous sections, but the tensor indices will now take the values (0, 1, 2, 3) while η_{ab} will denote the Minkowski metric. The generators now satisfy the equations

$$\begin{aligned}
&[M_{ab}, M_{cd}] = -\mathrm{i}\eta_{b[c}M_{ad]} + \mathrm{i}\eta_{a[c}M_{bd]}, \\
&[P_a, M_{bc}] = -\mathrm{i}\eta_{a[b}P_{c]},
\end{aligned} \tag{5.1}$$

---

[6] Closed differential systems of tensor-valued differential forms of any order are known as 'free differential algebras' (FDAs), see [29, 30].

and the corresponding MC 1-forms satisfy

$$d\mathcal{P}^a = -\mathcal{M}^a{}_c \wedge \mathcal{P}^c,$$
$$d\mathcal{M}^{ab} = -\mathcal{M}^{ac} \wedge \mathcal{M}_c{}^b. \tag{5.2}$$

Freezing the Lorentz freedom $\left(\mathcal{M}^a{}_b \to 0\right)$, the first-level extension results in the antisymmetric tensor 2-form $\mathcal{P}^a \wedge \mathcal{P}^b$. Using the same notation for the corresponding potential 1-form ($\mathcal{Z}^{[ab]}$) and reintroducing, as before, the Lorentz freedom, we arrive at

$$d\mathcal{Z}^{[ab]} = -\mathcal{M}^a{}_c \wedge \mathcal{Z}^{[cb]} - \mathcal{M}^b{}_c \wedge \mathcal{Z}^{[ac]} + \mathcal{P}^a \wedge \mathcal{P}^b, \tag{5.3}$$

which implies that the generators of translations no longer commute:

$$[P_a, P_b] = iZ_{[ab]}. \tag{5.4}$$

As there are no other extensions, the well-known result that there are no central extensions for the Poincaré group follows. This antisymmetric generator $Z_{[ab]}$ can be used to describe the motion of a relativistic particle in a constant electromagnetic field.

The calculation of the second level extensions results in 20 closed non-trivial 2-forms which can be written as the components of the tensor[7]

$$2\mathcal{P}^a \wedge \mathcal{Z}^{[bc]} - \mathcal{P}^b \wedge \mathcal{Z}^{[ca]} - \mathcal{P}^c \wedge \mathcal{Z}^{[ab]}. \tag{5.5}$$

Again, introducing the second-level potential 1-form $\mathcal{Y}^{a[bc]}$ with the same symmetries as the above 2-form tensor and unfreezing the Lorentz freedom, we find

$$d\mathcal{Y}^{a[bc]} = -\mathcal{M}^a{}_s \wedge \mathcal{Y}^{s[bc]} - \mathcal{M}^b{}_s \wedge \mathcal{Y}^{a[sc]} - \mathcal{M}^c{}_s \wedge \mathcal{Y}^{a[bs]}$$
$$+ 2\mathcal{P}^a \wedge \mathcal{Z}^{[bc]} - \mathcal{P}^b \wedge \mathcal{Z}^{[ca]} - \mathcal{P}^c \wedge \mathcal{Z}^{[ab]}. \tag{5.6}$$

As in the Galilei case, the generators of first and second level extensions now couple non-trivially to each other

$$[P_a, Z_{[bc]}] = 2iY_{a[bc]} - iY_{b[ca]} - iY_{c[ab]}. \tag{5.7}$$

Finally, the CE cohomology at higher degrees gives simple non-trivial results. For degree 3 there are four closed 3-forms which are the components of the pseudovector

$$\epsilon_{abcd}\mathcal{P}^b \wedge \mathcal{P}^c \wedge \mathcal{P}^d, \tag{5.8}$$

while at degree 4 there is a single pseudoscalar 4-form: $\epsilon_{abcd}\mathcal{P}^a \wedge \mathcal{P}^b \wedge \mathcal{P}^c \wedge \mathcal{P}^d$.

Acknowledgments

---

[7] This tensor is antisymmetric in $[bc]$ and its totally antisymmetric part vanishes. This leads to four identities, $\epsilon_{abcd}\mathcal{P}^b \wedge \mathcal{Z}^{[cd]} = 0$, leaving $4 \times 6 - 4 = 20$ independent components.

Appendix

The calculations in this paper are done using the Mathematica package EDC [26] and are contained in the Mathematica notebooks GalileiSequentialExtensions.nb and PoincareSequentialExtensions.nb, which are available from the authors. The calculations can be broken down into the following five distinct steps (results given for the Galilei case):

Step 1. Set $\mathcal{M}^{ab} = 0$ and construct all possible 2-forms from the set of 'translations' $\{\mathcal{H}, \mathcal{K}^a, \mathcal{P}^a\}$, obtaining a set of 21 2-forms. Find those combinations of these 2-forms that are closed and not trivial (cannot be written as d[1-form]). This leads to the following 12 2-forms:

$$\{\mathcal{H} \wedge \mathcal{P}^1, \mathcal{H} \wedge \mathcal{P}^2, \mathcal{H} \wedge \mathcal{P}^3, \mathcal{K}^1 \wedge \mathcal{K}^2, \mathcal{K}^1 \wedge \mathcal{K}^3, \mathcal{K}^2 \wedge \mathcal{K}^3, \mathcal{K}^1 \wedge \mathcal{P}^1, \mathcal{K}^2 \wedge \mathcal{P}^2, \mathcal{K}^1 \wedge \mathcal{P}^2$$
$$+ \mathcal{K}^2 \wedge \mathcal{P}^1, \mathcal{K}^1 \wedge \mathcal{P}^3 + \mathcal{K}^3 \wedge \mathcal{P}^1, \mathcal{K}^2 \wedge \mathcal{P}^3 + \mathcal{K}^3 \wedge \mathcal{P}^2, \mathcal{K}^3 \wedge \mathcal{P}^3\}.$$

Step 2. Group these 12 2-forms as the components of a vector $\mathcal{H} \wedge \mathcal{P}^a$, an antisymmetric tensor $\mathcal{K}^a \wedge \mathcal{K}^b$ and a symmetric tensor $\mathcal{K}^a \wedge \mathcal{P}^b + \mathcal{K}^b \wedge \mathcal{P}^a$. This step is the only non-algorithmic part of the procedure and can be non-trivial.

Step 3. Introduce the rotations and verify that these 2-forms remain 'closed' if we replace d by the 'covariant' exterior derivative $d + \mathcal{M}\wedge$.

Step 4. Now, if a tensor-valued 2-form $\mathcal{F}^a$ satisfies the equation[8] $d\mathcal{F}^a + \mathcal{M}^a{}_b \wedge \mathcal{F}^b = 0$, then there exists a corresponding tensor-valued 'potential' 1-form $\mathcal{Z}^a$ satisfying

$$d\mathcal{Z}^a + \mathcal{M}^a{}_b \wedge \mathcal{Z}^b = \mathcal{F}^a. \tag{A.1}$$

Introduce notation for the components of these 1-form potentials and write the component form of these equations. Verify consistency.

Step 5. Combine with the original algebra to obtain the extended algebra of MC 1-forms given in (2.14). Translate this algebra to the algebra of commutators of the corresponding generators.

The entire procedure of five steps is then repeated for the extended set of 2-forms (second level extension). The space of 2-forms now consists of 171 elements, of which 33 are found to be closed, non-trivial and not included in the first extension. They are the components of the tensors given in section 3.

Finally, the first two steps for the first-level 3-form and 4-form cohomology are also given. We find 18 closed non-trivial 3-forms and 19 closed non-trivial 4-forms which can be grouped as the components of the tensors given in section 4. Steps 3 and 4 (requiring now 2- or 3-form potentials) are straightforward.

References

[1] Bacry H and Levy-Leblond J 1968 Possible kinematics *J. Math. Phys.* **9** 1605
　　Bacry H and Nuyts J 1986 Classification of ten-dimensional kinematical groups with space isotropy *J. Math. Phys.* **27** 2455
[2] Bargmann V 1954 On unitary ray representations of continuous groups *Ann. Math.* **59** 1
[3] Lévy-Leblond J-M 1972 Galilei group and Galilean invariance *Group Theory and Applications* vol II, ed E M Loebl (New York: Academic) p 222
[4] Ballesteros A, Gadella M and del Olmo M 1992 Moyal quantization of 2 + 1 dimensional Galilean systems *J. Math. Phys.* **33** 3379

---

[8] The equations are written when $\mathcal{F}^a$ is a vector. For higher rank tensors additional $\mathcal{M}\wedge$ terms are needed—see equations (2.14) and (3.3).

 [5] Brihaye Y, Gonera C, Giller S and Kosiński P 1995 Galilean invariance in 2+1 dimensions arXiv:hep-th/9503046
 [6] Lukierski J, Stichel P C and Zakrzewski W J 1997 Galilean-invariant (2+1)-dimensional models with a Chern–Simons-like term and $D = 2$ noncommutative geometry *Ann. Phys.* **260** 224 (arXiv:hep-th/9612017)
 [7] Duval C and Horvathy P A 2000 The 'Peierls substitution' and the exotic Galilei group *Phys. Lett.* B **479** 284 (arXiv:hep-th/0002233)
     Duval C and Horvathy P A 2001 Exotic Galilean symmetry in the non-commutative plane, and the Hall effect *J. Phys. A: Math. Gen.* **34** 10097 (arXiv:hep-th/0106089)
 [8] Jackiw R and Nair V P 2000 Anyon spin and the exotic central extension of the planar Galilei group *Phys. Lett.* B **480** 237 (arXiv:hep-th/0003130)
 [9] Horvathy P A and Plyushchay M S 2002 Non-relativistic anyons, exotic Galilean symmetry and noncommutative plane *J. High Energy Phys.* JHEP06(2002)033 (arXiv:hep-th/0201228)
[10] Horvathy P A 2006 Non-commutative mechanics, in mathematical and in condensed matter physics arXiv:cond-mat/0609571
[11] Lévy-Leblond J-M 1969 Group-theoretical foundations of classical mechanics: the Lagrangian Gauge problem *Commun. Math. Phys.* **12** 64
[12] Aldaya V, de Azcarraga J A and Tucker R W 1986 Group cohomology approach to the quantum equivalence principle *J. Geom. Phys.* **3** 303
[13] Gadella M, Martin M A, Nieto L and del Olmo M 1991 The Stratonovich–Weyl correspondence for one-dimensional kinematical groups *J. Math. Phys.* **32** 1182
[14] Cangemi D and Jackiw R 1992 Gauge invariant formulations of lineal gravity *Phys. Rev. Lett.* **69** 233 (arXiv:hep-th/9203056)
[15] Bacry H, Combe P and Richard J L 1970 Group-theoretical analysis of elementary particles in an external electromagnetic field: 1. The relativistic particle in a constant and uniform field *Nuovo Cimento* A **67** 267
[16] Schrader R 1972 The Maxwell group and the quantum theory of particles in classical homogeneous electromagnetic fields *Fortschr. Phys.* **20** 701
[17] Beckers J and Hussin V 1983 Minimal electromagnetic coupling schemes: ii. Relativistic and nonrelativistic Maxwell groups *J. Math. Phys.* **24** 1295
[18] Soroka D V and Soroka V A 2005 Tensor extension of the Poincaré algebra *Phys. Lett.* B **607** 302 (arXiv:hep-th/0410012)
[19] Galperin A S, Ivanov E A, Ogievetsky V I and Sokatchev E 1988 Gauge field geometry from complex and harmonic analyticities. I. Kähler and self-dual Yang–Mills cases *Ann. Phys.* **185** 1
[20] Galperin A S, Ivanov E A, Ogievetsky V I and Sokatchev E 1988 Gauge field geometry from complex and harmonic analyticities. II. Hyper-Kähler *Ann. Phys.* **185** 22
[21] de Azcarraga J A and Izquierdo J M 1995 *Lie Groups, Lie Algebras, Cohomology and Some Applications in Physics* (Cambridge: Cambridge University Press)
[22] Gomis J and Ooguri H 2001 Non-relativistic closed string theory *J. Math. Phys.* **42** 3127 (arXiv:hep-th/0009181)
[23] Danielsson U H, Guijosa A and Kruczenski M 2000 IIA/B, wound and wrapped *J. High Energy Phys.* JHEP10(2000)020 (arXiv:hep-th/0009182)
[24] Gomis J, Gomis J and Kamimura K 2005 Non-relativistic superstrings: a new soluble sector of AdS(5) x S**5 arXiv:hep-th/0507036
[25] Brugues J, Curtright T, Gomis J and Mezincescu L 2004 Non-relativistic strings and branes as non-linear realizations of Galilei groups *Phys. Lett.* B **594** 227 (arXiv:hep-th/0404175)
[26] Bonanos S Exterior differential calculus, Mathematica enhancement, http://www.inp.demokritos.gr/~/sbonano/EDC/
[27] Coleman S R, Wess J and Zumino B 1969 Structure of phenomenological Lagrangians: 1 *Phys. Rev.* **177** 2239
     Callan C G, Coleman S R, Wess J and Zumino B 1969 Structure of phenomenological Lagrangians: 2 *Phys. Rev.* **177** 2247
[28] Gauntlett J P, Gomis J and Townsend P K 1990 Particle actions as Wess–Zumino terms for space-time (super) symmetry groups *Phys. Lett.* B **249** 255
[29] D'Auria R and Fre P 1982 Geometric supergravity in $D = 11$ dimensions and its hidden suopergroup *Nucl. Phys.* B **201** 101, section 2
[30] Castellani L and Perotto A 1996 Free differential algebras: their use in field theory and dual formulation *Lett. Math. Phys.* **38** 321 (arXiv:hep-th/9509031)